New Exact Travelling Wave Solutions of the Discrete Sine-Gordon Equation

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In this paper, we explore more applications of the hyperbolic function approach, which was used to find new exact travelling wave solutions of nonlinear partial differential equations or coupled nonlinear partial differential equations (PDES), to special discrete nonlinear equations. Some exact travelling wave solution of the discrete sine-Gordon equation are obtained in terms of hyperbolic function approach. – PACS numbers: 05.45.Yv, 02.30.Jr, 02.30.Ik

Key words: Discrete Sine-Gordon Equation; Exact Travelling Wave Solution; Hyperbolic Function Approach.

1. Introduction

Soliton dynamics in spatially nonlinear systems plays a crucial role in the modelling of many phenomena in different fields, ranging form condensed matter and biophysics to mechanical engineering. One also encounters such systems in numerical simulation of soliton dynamics in high energy physics where they arise as approximations of continuum models. Unlike difference equations which are fully discretized, differential-difference equations (DDES) are semi-discretized, with some (or all) of their spacial variables discretized, while time is usually kept continuous. A wealth of information about integrable differential-difference equations (DDES) can be found in papers by Suris [1 – 4], and his book [5], in progress. Suris and others have found many spatially discrete nonlinear models, such as the Volterra lattice models [1], the famous Toda lattice equations [6], the discrete MKdV equation [7], the discrete sine-Gordon equation [8], etc.

The investigation of the exact solutions of nonlinear DDES plays an important role in the study of nonlinear physical phenomena, and gradually becomes one of the most important and significant tasks. However, to our knowledge, less work has been done to investigate exact solutions of DDES. In contrast, a vast variety of methods has been developed to obtain exact solutions to a given nonlinear partial differential equation (NPDE), such as the inverse scattering method [9], the Bäcklund transformation [10], the homogenous balance method [11 – 14], the tanh method [15 – 17], the Jacobian elliptic function method [18 – 20], and the multilinear variable separation approach [21 – 24] et al. But as far as we can see, to extend the above methods to a differential difference equation (DDES) is rather difficult, though Qian and Lou [25] have successfully extended the multilinear variable separation to a special differential difference equation. More recently, Baldwin et al. [26] presented an algorithm to find exact travelling wave solutions of differential-difference equations in terms of tanh function and have found kink-type solutions in many spatially discrete nonlinear models. Explicit travelling wave solutions, which are widely encountered in science to model physical systems, are of great importance. In this paper, we look for further travelling solitary wave solutions of the discrete sine-Gordon equation [8]

$$\frac{\mathrm{d}u_{n+1}}{\mathrm{d}t} - \frac{\mathrm{d}u_n}{\mathrm{d}t} = \sin(u_{n+1} + u_n), \tag{1}$$

whose space is discrete, with lattice label n, and time is continuous, by means of the hyperbolic function approach.

Our paper is organized as follows. In Sect. 2, the detailed hyperbolic function method for DDES is given. In Sect. 3, we study the discrete sine-Gordon equation to illustrate the method. The last Section contains a short summary and discussion.

2. Hyperbolic Function Method for Nonlinear DDES

In this section we outline the hyperbolic function method for computation of closed form tanh-solutions to DDES. Consider a system of M polynomial DDES

$$\triangle(u_{n+p_1}(x),\dots,u_{n+p_k}(x),\dots,u'_{n+p_1}(x),\dots,\\ u'_{n+p_k}(x),\dots,u^{(r)}_{n+p_1}(x),\dots,u^{(r)}_{n+p_k}(x))=0, \quad (2)$$

where the dependent variable $\boldsymbol{u}$ has M components u_i, the continuous variable $\boldsymbol{x}$ has N components x_i, the discrete variable $\boldsymbol{n}$ has Q components n_j, the k shift vectors are $\boldsymbol{P}_i$, and $\boldsymbol{u}^{(r)}\boldsymbol{(x)}$ denotes the collection of mixed derivatives of order r.

2.1. Tanh Function Method

We introduce the hyperbolic tangent

$$T_n=\tanh\xi_n, \quad (3)$$

where

$$\xi_n=\sum_{i=1}^{Q}\mathrm{d}_i n_i+\sum_{j=1}^{N}c_j x_j+\zeta. \quad (4)$$

The coefficients $c_1,c_2,\dots,c_N,\mathrm{d}_1,\mathrm{d}_2,\dots,\mathrm{d}_Q$ and the phase ζ are constants. Repeatedly applying the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x_j}=\frac{\partial\xi_n}{\partial x_j}\frac{\mathrm{d}T_n}{\mathrm{d}\xi_n}\frac{\mathrm{d}}{\mathrm{d}T_n}=c_j(1-T_n^2)\frac{\mathrm{d}}{\mathrm{d}T_n}, \quad (5)$$

transforms (2) into

$$\triangle(U_{n+p_1}(T_n),\dots,U_{n+p_k}(T_n),\dots,\\ U'_{n+p_1}(T_n),\dots,U'_{n+p_k}(T_n),\dots,\\ U^{(r)}_{n+p_1}(T_n),\dots,U^{(r)}_{n+p_k}(T_n))=0. \quad (6)$$

Using the identity

$$\tanh(x+y)=\frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)}, \quad (7)$$

for any $s(s=1,\cdots,k)$, we can write

$$T_{n+p_s}=\frac{T_n+\tanh(\varphi_s)}{1+T_n\tanh(\varphi_s)}, \quad (8)$$

where

$$\varphi_s=p_{s1}d_1+p_{s2}d_2+\cdots+p_{sQ}d_Q, \quad (9)$$

and p_{sj} is the $j-$th component of shift vector $\boldsymbol{P}_s$.

Seeking solutions of the form

$$U_n(T_n)=\sum_{i=0}^{m}a_iT_n^i, \quad (10)$$

where a_i are constants to be determined and m is fixed by balancing the linear term of the highest order derivative with the highest order nonlinear term in (6), we obtain

$$U_{n+p_s}(T_n)=\sum_{i=0}^{m}a_iT_{n+p_s}^i=\sum_{i=0}^{m}a_i\left[\frac{T_n+\tanh(\varphi_s)}{1+T_n\tanh(\varphi_s)}\right]^i, \quad (11)$$

with φ_s from (9). Substituting (10) with (11) into (6), with the help of software *Maple*, clearing the denominator and eliminating all the coefficients of the powers of T_n, yields a series of algebraic equations, from which the parameters a_i and c_j are explicitly determined.

2.2. Coth Function Method

Along with the above idea, if the hyperbolic tangent function T_n in (3) is replaced by coth-function, namely,

$$T_n=\coth\xi_n, \quad (12)$$

with ξ_n defined in (4), the chain rule has still the form of (5).

Using the identity

$$\coth(x+y)=\frac{1+\coth(x)\coth(y)}{\coth(x)+\coth(y)}, \quad (13)$$

for any $s(s=1,\cdots,k)$, we can write

$$T_{n+p_s}=\frac{1+T_n\coth(\varphi_s)}{T_n+\coth(\varphi_s)}, \quad (14)$$

where φ_s satisfies (9).

We seek solutions of the form

$$U_n(T_n)=\sum_{i=0}^{m}a_iT_n^i, \quad (15)$$

where the a_i are constants to be determined and m is fixed by balancing the linear term of the highest order derivative with the highest order nonlinear term in (6). Then

$$U_{n+p_s}(T_n) = \sum_{i=0}^{m} a_i T_{n+p_s}^i = \sum_{i=0}^{m} a_i \left[\frac{1+T_n\coth(\varphi_s)}{T_n+\coth(\varphi_s)}\right]^i, \quad (16)$$

with φ_s from (9). Substituting (15) with (16) into (6), with the help of software *Maple*, clearing the denominator and eliminating all the coefficients of the powers of T_n, yields a series of algebraic equations, from which the parameters a_i and c_j are explicitly determined.

3. The Discrete Sine-Gordon Equation

Now we apply the method described in Sect. 2 to the discrete sine-Gordon equation (1), and give some exact travelling wave solutions.

Because it is difficult to apply directly the method to (1), we consider the transformations

$$u_n = 2\arctan v_n, \quad u_{n+1} = 2\arctan v_{n+1}, \quad (17)$$

and hence have

$$\frac{du_n}{dt} = \frac{2}{1+v_n^2}\frac{dv_n}{dt}, \qquad \frac{du_{n+1}}{dt} = \frac{2}{1+v_{n+1}^2}\frac{dv_{n+1}}{dt},$$
$$\sin u_n = \frac{2v_n}{1+v_n^2}, \qquad \cos u_n = \frac{1-v_n^2}{1+v_n^2}, \quad (18)$$
$$\sin u_{n+1} = \frac{2v_{n+1}}{1+v_{n+1}^2}, \quad \cos u_n = \frac{1-v_{n+1}^2}{1+v_{n+1}^2},$$

$$\sin(u_{n+1}+u_n) = sinu_{n+1}\cos u_n + \cos u_{n+1}\sin u_n = \frac{2v_{n+1}(1-v_n^2)+2v_n(1-v_{n+1}^2)}{(1+v_n^2)(1+v_{n+1}^2)}. \quad (19)$$

Substituting (18) and (19) into (1), (1) is reduced to a polynomial-type equation

$$(1+v_n^2)\frac{dv_{n+1}}{dt} - (1+v_{n+1}^2)\frac{dv_n}{dt} - v_n(1-v_{n+1}^2) - v_{n+1}(1-v_n^2) = 0. \quad (20)$$

3.1. Tanh Function Solution

Our main destination to find tanh-type solution of (20), so we suppose $v_n(n,t) \equiv v_n$, $v_{n+1}(n,t) \equiv v_{n+1}$ possess ansatzs as $v_n = \phi_n(T_n)$, $v_{n+1} = \phi_{n+1}(T_n)$, where $T_n = \tanh(kn+ct+\zeta)$. It is important to note that v_{n+1} is a function of T_n and not T_{n+1}. By repeatedly applying the chain rule (5), one gets

$$c(1-T_n^2)\left[(1+\phi_n^2)\frac{d\phi_{n+1}}{dT_n} - (1+\phi_{n+1}^2)\frac{d\phi_n}{dT_n}\right] - \phi_n(1-\phi_{n+1}^2) - \phi_{n+1}(1-\phi_n^2) = 0. \quad (21)$$

We expand the solution of (21) in the form of (10) and (11). In this case, $p_s = 1$, $d_1 = k$, $c_1 = c$. Balancing the linear term of the highest order derivative with the highest order nonlinear term in (21), we determined $m = 1$. We have

$$\phi_n(T_n) = a_0 + a_1 T_n, \qquad \phi_{n+1}(T_n) = a_0 + a_1\frac{T_n+\tanh(k)}{1+T_n\tanh(k)}. \quad (22)$$

Substituting (22) into (21), clearing the denominator and eliminating all the coefficients of the powers of T_n, yields the following algebraic equations

$$\begin{aligned}
&(a_1^3 + 2ca_0a_1^2)\tanh(k) + (ca_1^3 + ca_1a_0^2 + a_0a_1^2 + ca_1)\tanh^2(k) = 0,\\
&6a_0a_1^2 + (9a_1a_0^2 + 3a_1^3 - 3a_1)\tanh(k) + (4a_0a_1^2 + 2a_0^3 - 2a_0)\tanh^2(k) = 0,\\
&2a_1^3 + (6a_0a_1^2 + 2ca_1^3 + 2ca_1a_0^2 + 2ca_1)\tanh(k) + (3a_1a_0^2 + 2a_0^3 - 2a_0)\tanh^2(k) = 0,\\
&2a_0^3 - 2a_0 + (3a_1a_0^2 - 2ca_0a_1^2 - a_1)\tanh(k) + (a_0a_1^2 - ca_1a_0^2 - ca_1^3 - ca_1)\tanh^2(k) = 0,\\
&6a_1a_0^2 - 2a_1 + (6a_0a_1^2 - 2ca_1a_0^2 - 2ca_1^3 - 2ca_1 + 4a_0^3 - 4a_0)\tanh(k) + (3a_1a_0^2 - 4ca_0a_1^2 + a_1^3 - a_1)\tanh^2(k) = 0.
\end{aligned} \quad (23)$$

We solve the above system by using *Maple*, and obtain solutions, namely,

$$a_0 = 0, \quad a_1 = 1, \quad c = -\frac{1}{2\tanh(k)}, \qquad a_0 = 0, \quad a_1 = -1, \quad c = -\frac{1}{2\tanh(k)}, \quad (24)$$

where k is an arbitrary constant. According to (17), (22) and (24), we finally find an exact travelling wave solution of the discrete sine-Gordon equation (1)

$$u_n = 2\arctan\left\{\pm\tanh\left[kn - \frac{1}{2\tanh(k)}t + \zeta\right]\right\}. \quad (25)$$

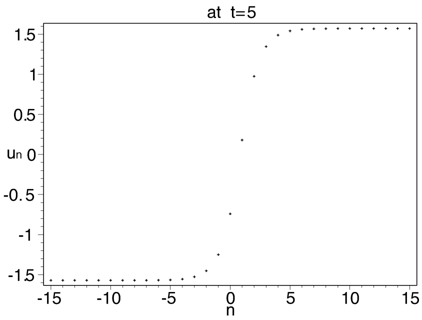

Fig. 1. The (+) branch of the kink-type solitary wave solution (25) at time $t = 5$ for the parameters $k = 0.5$, $\zeta = 5$.

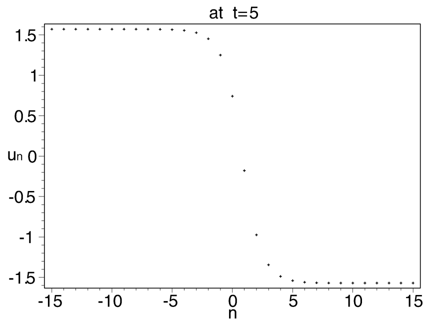

Fig. 2. The (–) branch of the anti-kink-type solitary wave solution (25), same parameters $t = 5$, $k = 0.5$, $\zeta = 5$.

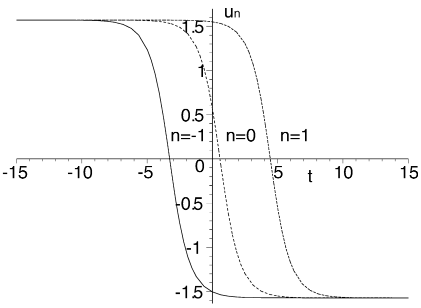

Fig. 3. The t-asymptotic form of the solution (25), (+) branch, parameters $k = 2$, $\zeta = 0.3$.

It is well known that soliton solutions are interesting and physical relevance. Here we take the solution (25) as an example to further analyze its properties by some figures. The solution (25) are kink-type and anti-kink-type solitary wave solutions, with fixed time t, whose relative properties are shown in Figs. 1 and 2. The asymptotic properties of the solution (25) with different lattice labels n are plotted in Figs. 3 and 4.

3.2. Coth Function Solution

Suppose $v_n(n,t) \equiv v_n$, $v_{n+1}(n,t) \equiv v_{n+1}$ possesses ansatzs as $v_n = \phi_n(T_n)$, $v_{n+1} = \phi_{n+1}(T_n)$, where $T_n =$

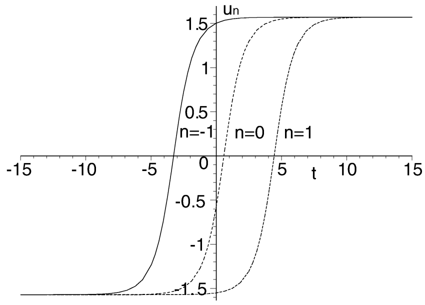

Fig. 4. The t-asymptotic form of the solution (25), (–) branch, parameters $k = 2$, $\zeta = 0.3$.

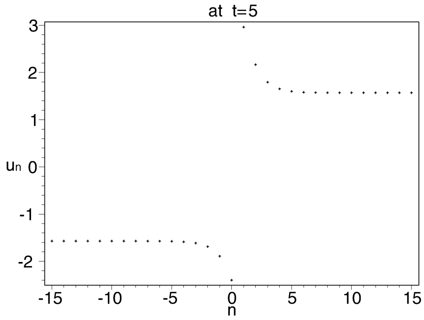

Fig. 5. Plot of the (+) branch solution of (29) for the parameters $t = 5$, $k = 0.5$, $\zeta = 5$.

$\coth(kn + ct + \zeta)$. By repeatedly applying the chain rule (5), one gets

$$\begin{aligned} &c(1-T_n^2)\left[(1+\phi_n^2)\frac{\mathrm{d}\phi_{n+1}}{\mathrm{d}T_n} - (1+\phi_{n+1}^2)\frac{\mathrm{d}\phi_n}{\mathrm{d}T_n}\right] \\ &-\phi_n(1-\phi_{n+1}^2) - \phi_{n+1}(1-\phi_n^2) = 0. \end{aligned} \tag{26}$$

In a similar way as in Sect. 3.1, we may choose the solution of (1) in the form

$$\begin{aligned} \phi_n(T_n) &= a_0 + a_1 T_n, \\ \phi_{n+1}(T_n) &= a_0 + a_1 \frac{1 + T_n \coth(k)}{T_n + \coth(k)}. \end{aligned} \tag{27}$$

Substituting (27) into (26), clearing the denominator and eliminating all the coefficients of the powers of T_n, which can be solved by using *Maple*, implies two solutions, i. e.,

$$\begin{aligned} &a_0 = 0, \quad a_1 = 1, \qquad c = -\frac{1}{2}\coth(k), \\ &a_0 = 0, \quad a_1 = -1, \quad c = -\frac{1}{2}\coth(k), \end{aligned} \tag{28}$$

where k is an arbitrary constant. According to (17), (27) and (28), we finally find an exact travelling wave

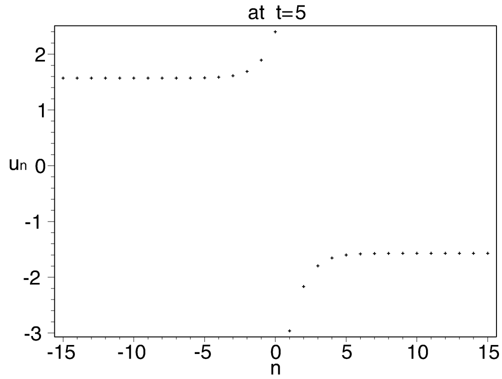

Fig. 6. Plot of the (–) branch solution of (29) with the parameters $t = 5$, $k = 0.5$, $\zeta = 5$.

solution of the discrete sine-Gordon equation (1)

$$u_n = 2\arctan\left\{\pm\coth\left[kn - \frac{1}{2}\coth(k)t + \zeta\right]\right\}. \quad (29)$$

As for illustration, the two solutions (29) for $t = 5$ are plotted in Figs. 5 and 6 choosing $k = 0.5$, $\zeta = 5$.

4. Summary and Discussion

In summary, we have solved the discrete sine-Gordon equation by using a hyperbolic function approach. As far as we know, this method has been successfully applied to solve other systems such as the celebrated Toda lattice model, the Ablowitz-Ladik lattice model and the Hybrid lattice system, and so on, and found the kink-shaped solutions [26]. However, this method fails to derive some bell-type and ring-type excitations. It is well known that the tanh method provides a straightforward and effective algorithm to obtain such particular solutions for a large NPDES. In recent years, much research work has been concentrated on various extensions to the improved tanh-method, the Jacobian elliptic function method and mapping approach for dealing with nonlinear partial differential physical systems. Interesting questions are: Can we obtain bell-type or other types excitations for the DDE system? Can the Jacobian function method be applied to some differential-difference lattice models? These are pending problems. The details for these cases will be investigated in our future works.

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